Research Statement

a. State of the Art and Objectives

The rigorous analysis and construction of quantum field models is by now a well established tradition, centering on two different approaches: the one based on Euclidean functional integrals, applied successfully to 2- and 3-dimensional models [27]; and a more recent trend focusing on operator-algebraic methods, employed in the construction of integrable models and controlled deformations thereof (see, for instance, Section 6 of the recent survey article [44], which also reviews the current state of the art of functional integral methods). However, so far both methods have failed to provide a rigorous construction of any of the four dimensional models which describe elementary particles as observed in accelerator experiments.

A more modest attempt at the issue has been provided by *perturbative algebraic quantum field* theory [5, 7], whose original aim was to give a sound mathematical basis to the formal perturbative renormalisation procedures employed by physicists, also envisaging the latter's extension to curved spacetimes [47, 6, 32]. However, it was soon realized [18, 19] that such an approach allowed one to see perturbative quantum field theory models as a *formal deformation* of the corresponding classical field theory models, in the sense that the product on the *-algebras of observables $\mathfrak{A}$ is a formal deformation of the corresponding operation in the classical Poisson algebra of observables $\mathfrak{A}_{cl}$ [2]

$$\mathfrak{A} = \mathfrak{A}_{cl}[[\hbar]] \ . \tag{1}$$

More precisely, the associative quantum product $F * G$ of observables $F, G \in \mathfrak{A}$ is a formal power series in $\hbar$ (seen as a formal deformation parameter instead of a fixed, positive real number)

$$F * G = \sum_{j=0}^{\infty} \hbar^j P_j(F, G) \ , \tag{2}$$

where the $\mathbb{C}[[\hbar]]$-bilinear maps P_j satisfy

$$F * G|_{\hbar=0} = P_0(F, G) = FG \ , \tag{3}$$

$$F * G - G * F = \hbar P_1(F, G) + O(\hbar^2) = i\hbar\{F, G\} + O(\hbar^2) \ , \tag{4}$$

with FG being the classical product and $\{F, G\}$ the Poisson bracket of the formal power series of any classical observables F, G. An associative product on $\mathfrak{A}$ satisfying (2)–(4) is called a **-product.* In the case of (relativistic) field theories over a d-dimensional space-time[1] $(\mathscr{M}, g)$, where elements of $\mathfrak{A}_{cl}$ ought to be functionals $F : \mathscr{Q} \ni \varphi \mapsto F(\varphi) \in \mathbb{C}$ on the classical field configuration space[2] $\mathscr{Q}$ and the dynamics is given by a first-order action functional

$$\mathscr{L}(f)(\varphi) = \int_{\mathscr{M}} \omega_{\mathscr{L}}(x, \varphi(x), \nabla\varphi(x))\mathrm{d}\mu_g(x) \ , \ f \in \mathscr{C}_c^{\infty}(\mathscr{M}) \tag{5}$$

with an hyperbolic (d-form valued) Euler-Lagrange operator $E(\mathscr{L})[\varphi]$ given by

$$\int_{\mathscr{M}} E(\mathscr{L})[\varphi]\vec{\varphi} = \mathscr{L}(f)^{(1)}[\varphi](\vec{\varphi}) \ , \ f \equiv 1 \text{ on supp } \vec{\varphi} \ , \tag{6}$$

a natural choice of Poisson bracket is provided by the *Peierls bracket* [16, 19]

$$\{F, G\}_{\mathscr{L}}(\varphi) = F^{(1)}[\varphi](\Delta_{\mathscr{L}}[\varphi]G^{(1)}[\varphi]) \ , \tag{7}$$

where $F^{(1)}[\varphi]$ denotes the functional derivative of F around the classical field configuration $\varphi \in \mathscr{Q}$ and $\Delta_{\mathscr{L}}[\varphi]$ is the causal propagator (= retarded minus advanced fundamental solution) of the linearized Euler-Lagrange operator $E'(\mathscr{L})[\varphi]$ around φ.

[1] That is, $(\mathscr{M}, g)$ is a globally hyperbolic Lorentzian manifold, whose volume element is denoted by $\mathrm{d}\mu_g$.

[2] Typically, $\mathscr{Q}$ is a subset of the space of smooth sections of some fibre bundle over the space-time manifold $\mathscr{M}$.

The deliberate a priori disregard of convergence issues for (2) is a necessity born of the unfortunate fact that the perturbative expressions for observable quantities in quantum field theory are usually divergent when the formal deformation parameter is given by the coupling constant. Condition (3) expresses in which sense $F * G$ is a deformation of FG, whereas condition (4) expresses the Dirac correspondence principle between quantum commutators and classical Poisson brackets. Conversely, if (2) is to define an associative product in $\mathfrak{A}$ such that P_0 is commutative, then $-i(P_1(F,G) - P_1(G,F))$ must define a $\mathbb{C}[[\hbar]]$-linear Lie bracket in $\mathfrak{A}$ which satisfies the Leibniz rule with respect to P_0. Although the physical justification of seeing the evaluation at $\hbar = 0$ as a "classical limit" is the target of heated debates on the related question of the origin of classicality in Quantum Physics [30, 26], the above "deformation paradigm" has produced a plethora of deep mathematical results, among which we can quote the works of Kontsevich [34], Fedosov [21] and Rieffel [39] on the existence of such deformations in different contexts (see [15, 48] for a more comprehensive list of references).

On the physical side, the realization of perturbative quantum field theory as a formal deformation of classical field theory is formally achieved by the so called *background field method* [16, 40, 49], where one understands the formal perturbative series in the coupling constant as a "stationary phase expansion" around a given field configuration and groups together all terms in the series with the same order in $\hbar$. However, one usually obtains an infinite number of terms of all orders in the coupling constant by doing so, therefore one would have in principle to deal with the convergence of each individual term in the $\hbar$ power series (called the *loop expansion* in the physical literature).

The above considerations give weight to the idea that perturbative quantum field theory should be a formal deformation of classical field theory in the sense of (1)–(4). However, any honest attempt at realizing this idea with the methods of perturbative algebraic quantum field theory faces the following challenges:

(i) What is the precise analog of $\mathfrak{A}_{cl}$ in field theory? That is, one must find a sufficiently well behaved Poisson algebra of functionals over $\mathscr{Q}$;

(ii) In which sense can $\mathscr{Q}$ be understood as an infinite dimensional Poisson manifold? That is, how is the geometry of $\mathscr{Q}$ is encoded in the structure of the Poisson algebra $\mathfrak{A}_{cl}$?

(iii) So far, the deformation picture provided by perturbative algebraic quantum field theory has only been fully realized around a *single* classical field configuration – namely, in the case of fields valued in vector bundles, the zero section. Can it be done *simultaneously* around *all* classical field configurations in $\mathscr{Q}$, so that we are indeed deforming $\mathfrak{A}_{cl}$?

(iv) In perturbative algebraic quantum field theory, the primary deformation parameter is not $\hbar$, but rather the coupling constant adjusting the strength of the nonlinearity in $E(\mathscr{L})$. Can we rephrase or strengthen this formalism so that the formal character of the coupling constant *is removed at each order in* $\hbar$ and therefore the latter remains as the only formal parameter?

(v) No address has been made to the other crucial structure of perturbative algebraic quantum field theory, namely, what states can be constructed? Can we apply our results to more practical research areas as in condensed matter?

(vi) Quantum physics and General Relativity seems to be yet far from each other, in spite of many efforts in the last decades in the areas of String Theory and Loop Quantum Gravity. A more traditional way at looking at their merging is to try to see perturbative quantum gravity as a locally covariant perturbative algebraic quantum field theory. Is this aim fruitful? Can one see how it compares with the practice in cosmology? Would it be possible to find the way to experimentally test the possible predicitons?

The objective of the present project is to address questions (i)–(vi) for general (relativistic) field theories in a thorough and mathematically precise manner, with special attention to the unravelling of new mathematical structures in the process.

b. Methodology

In the case of a scalar field, questions (i) and (ii) have been systematically addressed in recent work [10, 11, 12], aiming at an algebraic formulation for classical field theory where observables (i.e. functionals), as opposed to field configurations, constitute the central concept, following the premise advocated by Haag [28] in quantum field theory. In particular, one does not impose any equations of motion whatsoever to subsets of $\mathscr{Q}$, but rather imposes them on $\mathfrak{A}_{cl}$ (the so-called *off-shell algebra* of classical observables) by quotienting this algebra modulo the Poisson ideal $\mathscr{J}_{\mathscr{L}}$ generated by the Euler-Lagrange operator $E(\mathscr{L})$ (called the *on-shell ideal* of $\mathscr{L}$), very much in the spirit of algebraic geometry. Parts of the framework laid down therein have been extended to fermion fields [37], gauge fields and gravity [23, 38], and membranes [1]. This framework constitutes our conceptual basis for dealing with classical field theory and their quantum deformations.

Now we shall survey some directions to be pursued during the development of the project, which constitute of three main lines:

(A) Completing the algebraic framework for classical field theory put forward in the series [10, 11, 12], seeking to answer questions (i) and (ii);

(B) Using this framework to extend perturbative algebraic quantum field theory so as to incorporate arbitrary classical background fields, seeking to answer questions (iii) and (iv).

(C) Question (v) needs results recently elaborated in [22] and are meant to be applied to the Bose-Einstein condensation for relativistic complex bosons in interactions, as a first attempt.

(D) An answer to the first question in (vi) has been recently given [9]. Upon this basis we are trying to seek answers for the rest.

Let us briefly describe each of these lines.

b.1. Line of investigation A – classical field theory

Of course, questions (i) and (ii) are intertwined and constitute stepping stones for addressing questions (iii) and (iv) in a mathematically satisfactory manner, but one readily sees they are also interesting in their own right. As explained below, neither of them are expected to have a trivial answer, even in the simplest cases.

b.1.1. Infinite dimensional geometry of the space of field configurations

Due to the distributional character of $\Delta_{\mathscr{L}}[\varphi]$, (7) is usually ill-defined for general, smooth functionals F, G, so $\mathfrak{A}_{cl}$ cannot be the whole algebra of differentiable functionals on $\mathscr{Q}$. Therefore, one imposes the following restrictions on the elements $F \in \mathfrak{A}_{cl}$:

(a) *Smoothness* – F should have functional derivatives $F^{(k)}[\varphi]$ of all orders $k \in \mathbb{N}$ at all $\varphi \in \mathscr{Q}$, defining kd-form-valued distributions with compact support on $\mathscr{M}^k$;

(b) *Support* – The *space-time support* of F

$$\text{supp } F = \{p \in \mathscr{M} | F(\varphi_1) \neq F(\varphi_2), \forall \varphi_1, \varphi_2 : \varphi_1(p) \neq \varphi_2(p)\}$$

is compact;

(c) *Singular structure* – the wave front set of $F^{(k)}[\varphi]$ satisfies $\mathrm{WF}(F^{(k)}[\varphi]) \cap (\overline{V}^k_+ \cup \overline{V}^k_-) = \varnothing$ for all $k \in \mathbb{N}$, $\varphi \in \mathscr{Q}$, where $\overline{V}_\pm = \{(p, \xi) \in T^*\mathscr{M} | \xi$ is future (resp. past) directed causal$\}$.

Functionals $F : \mathscr{Q} \to \mathbb{C}$ satisfying (a)–(c) are said to be *microcausal*. Such a definition encompasses the following subspaces of functionals:

- *Regular* functionals: $F^{(k)}[\varphi]$ is a smooth kd-form of compact support on $\mathscr{M}^k$ for all $k \in \mathbb{N}$, $\varphi \in \mathscr{Q}$. We denote the subalgebra of regular functionals by $\mathfrak{A}_0$;
- *Local* functionals: $F^{(1)}[\varphi]$ is a smooth d-form of compact support on $\mathscr{M}$ for all $\varphi \in \mathscr{Q}$ and supp $F^{(2)}[\varphi] \subset \Delta_2(\mathscr{M})$, where $\Delta_k(\mathscr{M}) = \{(p, \ldots, p) \in \mathscr{M}^k | p \in \mathscr{M}\}$ is the *small diagonal* of $\mathscr{M}^k$ for each $k \in \mathbb{N}$. We denote the subspace of local functionals by $\mathfrak{A}_{loc}$.

With the requirements (a)–(c), $\mathfrak{A}_{cl}$ does indeed become an *almost Poisson algebra* [15] when endowed with the Peierls bracket. If the elements of $\mathscr{Q}$ are sections of a fibre bundle $\pi : E \to \mathscr{M}$ whose typical fibre Q admits a flat connection (for instance, vector and affine bundles), then the Jacobi identity also holds for all $F, G, H \in \mathfrak{A}_{cl}$ [10]

$$J_{\mathscr{L}}(F, G, H) \doteq \{\{F, G\}_{\mathscr{L}}, H\}_{\mathscr{L}} + q\{\{G, H\}_{\mathscr{L}}, F\}_{\mathscr{L}} + q\{\{H, F\}_{\mathscr{L}}, G\}_{\mathscr{L}} = 0$$

and $(\mathfrak{A}_{cl}, \{\cdot, \cdot\}_{\mathscr{L}})$ is actually a Poisson algebra. More generally, the Jacobiator $J_{\mathscr{L}}(F, G, H)$ of the Peierls bracket $\{\cdot, \cdot\}_{\mathscr{L}}$ is an element of the on-shell ideal $\mathscr{J}_{\mathscr{L}}$ for all $F, G, H \in \mathfrak{A}_{cl}$, hence the quotient space $\mathfrak{A}_{cl}/\mathscr{J}_{\mathscr{L}}$ of on-shell functionals is always a Poisson algebra. On the other hand, the curvature of the typical fibre Q leads to a nonzero Jacobiator in the off-shell algebra $\mathfrak{A}_{cl}$. A very interesting question is what kind of infinite dimensional generalization of Poisson geometry such an almost Poisson algebra leads to, even in cases where the curvature of the typical fibre enjoys a good deal of homogeneity, such as Riemannian symmetric spaces.

Another important aspect of $\mathfrak{A}_{cl}$ is the fact that it is also a $\mathscr{C}^\infty$-*ring*: given functionals $F_1, \ldots, F_n \in \mathscr{E}_{cl}$ and a smooth map $\Phi : \mathbb{C}^n \cong \mathbb{R}^{2n} \to \mathbb{C}$, we have that $\Phi \circ (F_1, \ldots, F_n)$ is also an element of $\mathfrak{A}_{cl}$. This implies, among other things, that $\mathfrak{A}_{cl}$ distinguishes all field configurations of $\mathscr{Q}$ (i.e. given $\varphi_1 \neq \varphi_2 \in \mathscr{Q}$, there is a $F \in \mathfrak{A}_{cl}$ such that $F(\varphi_1) \neq F(\varphi_2)$) and can even be used to build functional partitions of unity [10]. This also opens the possibility of using methods from $\mathscr{C}^\infty$ algebraic geometry [33].

b.1.2. Dynamics of field theories with local gauge symmetry

As mentioned in the beginning of this Section, we do not impose any equations of motion on the classical field configuration space $\mathscr{Q}$. We can nevertheless analyse the efffect that small, *relative* perturbations of a given dynamics have on the elements of $\mathfrak{A}_{cl}$. Given an action functional $\mathscr{L}$ as in (5) with hyperbolic Euler-Lagrange operator $E(\mathscr{L})$ as in (6), one set the problem of finding solutions φ to the equation

$$E(\mathscr{L})[\varphi] = E(\mathscr{L})[\varphi_0] + \omega \tag{8}$$

in a compact region K of $\mathscr{M}$, which are close to a given $\varphi_0 \in \mathscr{Q}$ when the d-form ω is small on K. In particular, one may take $\omega = E(\mathscr{L}_0)[\varphi_0] - E(\mathscr{L})[\varphi_0]$, where $\mathscr{L}_0$ is a perturbation of $\mathscr{L}$. In this case $\varphi = \mathsf{m}(\varphi_0)$ formally defines at each φ_0 in a small neighbourhood $\mathscr{U} \subset \mathscr{Q}$ an intertwing map $\mathsf{m} : \mathscr{U} \to \mathscr{Q}$ of $E(\mathscr{L})$ and $E(\mathscr{L}')$ in K, that is, a smooth map satisfying

$$E(\mathscr{L})[\mathsf{m}(\varphi)](p) = E(\mathscr{L}_0)[\varphi](p)\ ,\ \forall p \in K\ ,\ \varphi \in \mathscr{U}\ . \tag{9}$$

This map determines how the elements of $\mathscr{U}$ are “scattered” by the perturbation. As such, we call m a *Møller map* from $\mathscr{L}_0$ to $\mathscr{L}$.

As a matter of fact, one can actually prove the existence[3] of solutions of (8) in a sufficiently small neighbourhood $\mathscr{U}$ of any $\varphi_0 \in \mathscr{Q}$ in the case of real scalar fields [11]. The strategy of proof consists in applying the Nash-Moser-Hrmander implicit function theorem to (8) supplied by refined energy estimates on the linearised Euler-Lagrange operator $E'(\mathscr{L})[\varphi]$. The local existence of Møller maps m then follows immediately, with a number of deep consequences. For instance, one can show that m is a *Poisson map* from the Poisson algebra $(\mathfrak{A}_{cl}, \{\cdot,\cdot\}_{\mathscr{L}})$ to the Poisson algebra $(\mathfrak{A}_{cl}, \{\cdot,\cdot\}_{\mathscr{L}_0})$, that is,

$$\{F \circ \mathsf{m}, G \circ \mathsf{m}\}_{\mathscr{L}_0}(\varphi) = \{F, G\}_{\mathscr{L}} \circ \mathsf{m}(\varphi)\ ,\ \forall F, G \in \mathfrak{A}_{cl}\ ,\ \varphi \in \mathscr{U}\ .$$

In particular, in the case $E(\mathscr{L}_0)[\varphi] = E'(\mathscr{L})[\varphi_0](\varphi - \varphi_0)$ is the linearisation of $E(\mathscr{L})$ around φ_0, one has that m linearises the Peierls bracket of $\mathscr{L}$ in $\mathscr{U}$.

The analysis sketched above can be extended to fields living in more general bundles, but much of it has to be largely overhauled when the Lagrangian $\mathscr{L}$ possesses local symmetries, for then the Euler-Lagrange operator $E(\mathscr{L})$ ceases to be hyperbolic in the usual sense. Nevertheless, the algebraic viewpoint employed here can be suitably extended in order to incorporate this scenario. In the case of pure Yang-Mills fields and gravity, this has been done in [23, 31, 38] by means of a version of the Batalin-Vilkoviskiĭ formalism adapted to our framework, which provides a homological resolution of the on-shell algebra $\mathfrak{A}_{cl}/\mathscr{J}_{\mathscr{L}}$ by adding extra field degrees of freedom while keeping track of the structure of the local symmetries of $\mathscr{L}$.

However, this formalism by itself has little to do with the hyperbolicity of the Euler-Lagrange equations, which is necessary even to define a Peierls bracket on $\mathfrak{A}_{cl}$ or suitable extensions thereof. This task has been accomplished so far only in a trial-and-error basis, by adding gauge-fixing Lagrange multipliers to $\mathscr{L}$ adapted to the model being studied. Finding a gauge-invariant definition of hyperbolicity for systems of partial differential equations, or even more explicit and systematic criteria for devising such gauge-fixing terms, constitutes a major gap in the mathematical literature.

A further step would be to make a local analysis of the gauge orbits in $\mathscr{Q}$. The goal is to employ this information together with the functional partitions of unity constructed in [10] to circumvent the Gribov-Singer phenomenon [43] by localizing the procedure of gauge fixing.

b.1.3. The principle of space-time descent

An important ingredient that has been missing so far in our dicussion is the issue of local covariance. Namely, given a field theoretical model, we want to know how the (almost) Poisson algebras $\mathfrak{A}_{cl}(\mathscr{M}, g)$ associated to each d-dimensional globally hyperbolic space-time $(\mathscr{M}, g)$ relate to each other. The relevance of this question lies in the fact that the physical content of a field theory does not lie within single algebras of observables, but

[3]Uniqueness can also be proven, provided that suitable boundary conditions are satisfied.

rather within the relations among them [28, 13].

We can make this question more precise in terms of the language of categories and functors, in terms of the so-called *principle of local covariance* [13, 7, 12]. Let $\mathbf{Loc}_d$ be the category whose objects are d-dimensional globally hyperbolic space-times $(\mathscr{M}, g)$ and whose arrows $\psi : (\mathscr{M}, g) \to (\mathscr{M}', g')$ are isometric embeddings $\psi\mathscr{M} \to \mathscr{M}'$ with globally hyperbolic range $(\psi(\mathscr{M}), g'|_{\psi(\mathscr{M})}) = (\psi(\mathscr{M}'), (\psi^{-1})^* g)$. Then, a *locally covariant classical field theory* is simply a covariant functor

$$\mathfrak{A}_{cl} : \begin{cases} (\mathscr{M}, g) & \mapsto \mathfrak{A}_{cl}(\mathscr{M}, g) \ , \\ (\psi : (\mathscr{M}, g) \to (\mathscr{M}', g')) & \mapsto (\mathfrak{A}_{cl}\psi : \mathfrak{A}_{cl}(\mathscr{M}, g) \to \mathfrak{A}_{cl}(\mathscr{M}', g')) \end{cases}$$

from $\mathbf{Loc}_d$ to the category of Poisson algebras and Poisson homomorphisms [7, 12]. More concretely, in the case of *real scalar* fields the fact that $\mathfrak{A}_{cl}(\mathscr{M}, g)$ is an algebra of functionals suggests the formula

$$\mathfrak{A}_{cl}\psi(F)(\varphi) = F(\varphi \circ \psi) \ ,$$

which clearly defines an algebra homomorphism since F is supposed to be insensitive to variations of its argument outside $\psi(\mathscr{M})$. It is even a Poisson map, provided that the Peierls brackets of $\mathfrak{A}_{cl}(\mathscr{M}, g)$ and $\mathfrak{A}_{cl}(\mathscr{M}', g')$ are respectively given by Lagrangians $\mathscr{L}, \mathscr{L}'$ satisfying

$$S_{\mathscr{L}'}(\psi_* f)(\varphi) = S_{\mathscr{L}}(f)(\psi^*\varphi) \ .$$

All is well and fine, apart from one crucial caveat: we have not discussed how the *domains* $\mathscr{Q}(\mathscr{M}, g)$ of the functionals $F \in (\mathscr{M}, g)$ relate to each other, which can be seen as (pure) *state spaces* for the algebras of observables $\mathfrak{A}_{cl}(\mathscr{M}, g)$. If we want to keep the freedom in restricting the domain of F (which even turns out to be necessary in the analysis of dynamics from an algebraic viewpoint [12]), we should weaken the relation $\mathscr{Q}(\mathscr{M}, g) = \psi^*\mathscr{Q}(\mathscr{M}', g') \doteq \{\psi^*\varphi | \varphi \in \mathscr{Q}(\mathscr{M}', g')\}$ to

$$\mathscr{Q}(\mathscr{M}, g) \subset \psi^*\mathscr{Q}(\mathscr{M}', g') \ .$$

To allow for simultaneous specification of the space-time and functional domains, it is convenient to replace $\mathbf{Loc}_d$ by another category $\mathscr{S}$, called a *localization category* over $\mathbf{Loc}_d$, whose objects are given by pairs $((\mathscr{M}, g), \mathscr{U})$ with $(\mathscr{M}, g)$ an object of $\mathbf{Loc}_d$ and $\mathscr{U}$ a subset of classical field configurations over $(\mathscr{M}, g)$, and whose arrows are given by $\psi_* = (\psi, (\psi^*)^{op}) : ((\mathscr{M}, g), \mathscr{U}) \to ((\mathscr{M}', g'), \mathscr{U}')$, where ψ is an arrow in $\mathbf{Loc}_d$ and $(\psi^*)^{op}$ is the opposite arrow to the pullback ψ^* of field configurations by ψ, seen as a map from $\mathscr{U}'$ *into* $\mathscr{U}$.

A further ingredient allows one to fully determine a locally covariant classical field theory by means of local data both at the level of space-times and field configurations. Namely, the category $\mathbf{Loc}_d$ admits a notion of *covering* of an object by a family of other objects "up to embeddings", which allows us to define *sheaves* over $\mathbf{Loc}_d$. Moreover, one can even "glue together" objects of $\mathscr{S}$ over a covering, provided certain compatibility conditions, called *descent conditions*, are satisfied. This shows that $\mathscr{S}$ constitutes what is called a *stack* over $\mathbf{Loc}_d$ [46].

This allows one to fully determine a locally covariant classical field theory by local data - this strengthening of the principle of local covariance is called the *principle of space-time descent* [12]. An extension of this principle to field theories with internal symmetries living in general fibre bundles should be rather straightforward, provided one replace arrows in $\mathbf{Loc}_d$ by bundle maps covering them and supplying the scalar descent conditions with bundle cocycle conditions. One powerful feature of stacks is that they behave well under quotients – indeed, one of the motivations for introducing stacks was the study of moduli spaces. In our case, an obvious application of this language is the study of the moduli space of solutions of, say, Yang-Mills equations modulo gauge transformations.

b.2. Line of investgation B – quantum field theory

We shall now outline a strategy to attack problems (iii) and (iv) using the algebraic framework for classical field theory proposed in [10, 11]. For simplicity, we consider only the case of real scalar fields.

b.2.1. Deformation quantisation of field theories

As discussed in the previous Section, the standard framework of perturbative quantum field theory provides a deformation quantisation of the corresponding classical field theoretical model around a single field configuration $\varphi_0 \in \mathscr{Q}$.

Let us first perform the quantisation of the linearised theory around φ_0. The first *-product we introduce is the so called *Weyl-Moyal* *-product:

$$(F * G)(\varphi_0) = e^{i\frac{\hbar}{2}\langle \Delta_{\mathscr{L}}[\varphi_0], D_1 \otimes D_2\rangle}(F(\varphi_1)G(\varphi_2))|_{\varphi_1=\varphi_2=\varphi_0} \ ,$$

where D_j stands for the functional derivative w.r.t. j-th argument, $j = 1, 2$. This product is well defined for $F, G \in \mathfrak{A}_0$. This comprises functionals such as the *Weyl unitaries*

$$W(\omega)(\varphi) = \exp\left(\int_{\mathscr{M}} \varphi(x)\omega(x)\right) \ , \omega \in \Gamma_c^\infty(\wedge^d T^*\mathscr{M} \to \mathscr{M}) \ ,$$

in which case we recover the *Weyl form of the canonical commutation relations*

$$(W(\omega_1) * W(\omega_2))(\varphi_0) = e^{\frac{i\hbar}{2}\Delta_{\mathscr{L}}[\varphi_0](\omega_1,\omega_2)} W(\omega_1 + \omega_2)(\varphi_0) \ , \forall \omega_1, \omega_2 \in \Gamma_c^\infty(\wedge^d T^*\mathscr{M} \to \mathscr{M}) \ ,$$

but obviously excludes local, nonlinear functionals such as $F(\varphi) = \int_{\mathscr{M}} \varphi^2(x)\omega(x)$. A more convenient *-product for quantum field theory is the *Wick-Voros *-product* associated to a choice of symmetric Hadamard propagator $\Delta^H_{\mathscr{L}}[\varphi_0]$ for $E'(\mathscr{L})[\varphi_0]$

$$(F *_H G)(\varphi_0) = e^{\hbar\langle \Delta^{+,H}_{\mathscr{L}}[\varphi_0], D_1 \otimes D_2\rangle}(F(\varphi_1)G(\varphi_2))|_{\varphi_1=\varphi_2=\varphi_0} \ ,$$

where $\Delta^{+,H}_{\mathscr{L}}[\varphi_0] = \frac{1}{2}\Delta^H_{\mathscr{L}}[\varphi_0] + \frac{i}{2}\Delta_{\mathscr{L}}[\varphi_0]$ is the *Wightman propagator* $\mathscr{E}'(\mathscr{L})[\varphi_0]$ associated to $\Delta^H_{\mathscr{L}}[\varphi_0]$. One should see $\Delta^{+,H}_{\mathscr{L}}[\varphi_0]$ as a microlocal remnant of the "positive frequency" part of $\Delta_{\mathscr{L}}[\varphi_0]$ [24, 25, 36]. In this case, $(F *_H G)(\varphi_0)$ is well defined as an element of $\mathbb{C}[[\hbar]]$ for all $F, G \in \mathfrak{A}_{cl}$ [18, 19]. The *-products $*$ and $*_H$ as one varies $\varphi_0 \in \mathscr{Q}$ and $\Delta^H_{\mathscr{L}}[\varphi_0]$ define "Weyl algebra bundles" over $\mathscr{Q}$, where each fibre has the form of a power series vector space

$$(\oplus_{k=0}^\infty \mathscr{E}_k)\,[[\hbar]]$$

with $\mathscr{E}_0 = \mathbb{C}$ and the spaces of distributions $\mathscr{E}_k \subset \mathscr{E}'(\wedge^{kd} T^*\mathscr{M}^k \to \mathscr{M}^k)$, $k \geq 1$ satisfying the property $\mathscr{E}_k \otimes \mathscr{E}_l \subset \mathscr{E}_{k+l}$ for all $k, l \geq 0$. Obviously, Taylor series of elements of both $\mathfrak{A}_0$ and $\mathfrak{A}_{cl}$ at $\varphi_0 \in \mathscr{Q}$ satisfy these requirements.

The *-product which describes the structure of the full, *interacting* quantum theory around a fixed background field $\varphi_0 \in \mathscr{Q}$ requires the input of perturbation theory. The approach to perturbative renormalisation employed in perturbative algebraic quantum field theory is the Epstein-Glaser approach [20, 42, 6]. In order to apply this renormalisation method in our context, on needs to perform a space-time cutoff of the nonlinearity of the Lagrangian $\mathscr{L}$ adapted to φ_0. The Taylor expansion of the action functional $\mathscr{L}(f)$ around φ_0 with cubic remainder $\mathscr{L}_{int}(f)$ yields

$$\mathscr{L}(f)(\varphi_0 + \vec{\varphi}) = \mathscr{L}(f)(\varphi_0) + S_{\mathscr{L}}(f)^{(1)}[\varphi_0](\vec{\varphi}) + \frac{1}{2}\mathscr{L}(f)^{(2)}[\varphi_0](\vec{\varphi}, \vec{\varphi}) + \mathscr{L}_{int}(f)(\varphi_0, \vec{\varphi}) \ .$$

The Euler-Lagrange (EL) operator $E(\mathscr{L})$ associated to $\mathscr{L}$ can then be written as

$$E(\mathscr{L})[\varphi_0 + \vec{\varphi}] = E(\mathscr{L})[\varphi_0] + E'(\mathscr{L})[\varphi_0]\vec{\varphi} + E(\mathscr{L}_{int})[\varphi_0, \varphi] \ ,$$

where $E(\mathscr{L})[\varphi_0]$ is called the *source term* of $E(\mathscr{L})$ at φ_0, $E'(\mathscr{L})[\varphi_0]\vec{\varphi}$ is the linearisation of $E(\mathscr{L})$ at φ_0 and $E(\mathscr{L}_{int})[\varphi_0, \varphi]$ is called the *interaction term* of $E(\mathscr{L})$ at φ_0. Given $h \in \mathscr{C}_c^\infty(\mathscr{M})$, the *h-cutoff* $\mathscr{L}|_h = \mathscr{L}|_h(f)(\varphi_0, \vec{\varphi})$ of $\mathscr{L}$ around φ_0 is defined by replacing $\mathscr{L}_{int}(f)(\varphi_0, \vec{\varphi})$ by

$$(\mathscr{L}|_h)_{int}(f)(\varphi_0, \vec{\varphi}) = \mathscr{L}_{int}(hf)(\varphi_0, \vec{\varphi}) \ .$$

The corresponding *h-cutoff* $E(\mathscr{L}|_h) = E(\mathscr{L}|_h)[\varphi_0, \vec{\varphi}]$ of $E(\mathscr{L})$ around φ_0, on its turn, is given by the Euler-Lagrange operator of $\mathscr{L}|_h$ with φ_0 fixed.

Now we are in position to introduce interactions at the quantum level. To wit, if $\Delta^{F,H}_{\mathscr{L}}[\varphi_0] = \Delta^{+,H}_{\mathscr{L}}[\varphi_0] + i\Delta^A_{\mathscr{L}}[\varphi_0]$ is the *Feynman propagator* of $E'(\mathscr{L})[\varphi_0]$ associated to $\Delta^H_{\mathscr{L}}[\varphi_0]$, one can formally write *Bogolyubov's formula* for the *retarded interacting functional* at φ_0

$$\mathsf{R}_{\mathscr{L}_{int}|_{\lambda h}}(F)(\varphi_0) = S(\mathscr{L}_{int}|_{\lambda h})^{*_H-1} *_H (S(\mathscr{L}_{int}|_{\lambda h}) \cdot_{T_H} F)(\varphi_0) \in \mathbb{C}[[\hbar, \lambda]]$$

associated to $F \in \mathfrak{A}_{cl}$, where

$$S(\mathscr{L}_{int}|_{\lambda h})(\varphi_0) = T_H \circ \exp \circ T_H^{-1} \mathscr{L}_{int}(\lambda h)(\varphi_0)$$

is *Bogolyubov's S-matrix* associated to $\mathscr{L}_{int}|_{\lambda h}$, $F \cdot_{T_H} G(\varphi_0) = T_H(T_H^{-1}F \cdot T_H^{-1}G)$ is the *time ordered product* of F, G, and

$$T_H F(\varphi_0) = e^{\hbar\langle \Delta^{T_H}_{\mathscr{L}}[\varphi_0], D_1 \otimes D_2\rangle} F(\varphi_1)G(\varphi_2)|_{\varphi_1=\varphi_2=\varphi_0}$$

is the *time ordering operator* at φ_0 associated to $\Delta^H_{\mathscr{L}}[\varphi_0]$. The time ordering operator $F \cdot_{T_H} G$ is not *a priori* well defined for all $F, G \in \mathfrak{A}_{cl}$. The key observation that allows us to solve this problem for F, G *local*, say,

$$F(\varphi) = \int_{\mathscr{M}} f(j^k\varphi)^*\omega \ , G(\varphi) = \int_{\mathscr{M}} f'(j^{k'}\varphi)^*\omega' \ , \ f, g \in \mathscr{C}^\infty_c(\mathscr{M}) \ , \ k, k' \in \mathbb{N} \ ,$$

is that $F \cdot_{T_H} G = F *_H G$ whenever there is a Cauchy hypersurface Σ in $(\mathscr{M}, g)$ such that supp f lies to the chronological future of Σ and supp f' lies to the chronologigal past of Σ. This defines $F \cdot_{T_H} G(\varphi_0)$ as a distribution in $f \otimes f'$ at each order in $\hbar$ up to the *small diagonal* $\Delta_2(\mathscr{M}) = \{(x, x) \in \mathscr{M}^2 | x \in \mathscr{M}\}$ of $\mathscr{M}^2$. Analysing the singular behaviour of $F \cdot_{T_H} G(\varphi_0)$ near $\Delta_2(\mathscr{M})$ allows one to find and classify all possible distribution extensions to the whole of $\mathscr{M}^2$ – in fact, the procedure allows one to define $\cdot_{T_H}$ as a bona fide commutative and associative product at φ_0 on the subalgebra of $\mathfrak{A}_{cl}$ generated by local functionals.

At this point, it is important to have a closer look at the *classical* (i.e. $\hbar^0$) part of $\mathsf{R}_{\mathscr{L}_{int}|_{\lambda h}}(F)$:

$$\mathsf{R}_{\mathscr{L}_{int}|_{\lambda h}}(F) \stackrel{\hbar \to 0}{\longrightarrow} F \circ \mathsf{r}_{\mathscr{L}|_{\lambda h}, \mathscr{L}|_{\lambda h} - \mathscr{L}_{int}|_{\lambda h}} \ .$$

One sees that the map $\mathsf{r} = \mathsf{r}_{\mathscr{L}|_{\lambda h}, \mathscr{L}|_{\lambda h} - \mathscr{L}_{int}|_{\lambda h}}$ defined by the above classical limit is a *retarded* Møller map, that is,

$$\begin{aligned} E(\mathscr{L}|_{\lambda h})[\varphi_0, \mathsf{r}(\varphi_0 + \vec{\varphi}) - \varphi_0] &= E(\mathscr{L})[\varphi_0] + E'(\mathscr{L})[\varphi_0]\vec{\varphi} \ , \\ \mathsf{r}(\varphi_0 + \vec{\varphi})(p) &= \varphi_0(p) + \vec{\varphi}(p) \ , \quad \forall p \notin J^+(\text{supp } h) \ . \end{aligned}$$

Recall from Subsubsection b.1.2 that the second condition formally implies that r is a *Poisson map*:

$$\{F \circ \mathsf{r}, G \circ \mathsf{r}\}_{\mathscr{L}|_{\lambda h} - \mathscr{L}_{int}|_{\lambda h}} = \{F, G\}_{\mathscr{L}|_{\lambda h}} \circ \mathsf{r} \ .$$

As argued in Subsubsection b.1.2, r exists *non-perturbatively* (i.e. exactly) as a smooth map defined in a small neighborhood $\mathscr{U} \ni \varphi_0$ [11], hence the Peierls bracket can be locally linearized by the change of field coordinates promoted by r. This allows us to introduce the following *semiclassical interacting *-product* at $\mathscr{U}$ [14, 17]

$$F \circledast_H G(\varphi) = [(F \circ \mathsf{r}) *_H (G \circ \mathsf{r})] \circ \mathsf{r}^{-1}(\varphi) \ ,$$

where $*_H$ uses $\Delta^{+,H}_{\mathscr{L}}[\varphi_0]$.

One can show that the pullback of the standard functional derivative D on $\mathscr{U}$ (seen as a *covariant* derivative) by r is given by $\mathsf{r}^*D|_\varphi = (D + \Gamma^R)|_{\mathsf{r}(\varphi)}$, where the Christoffel symbol Γ^R is given by

$$\Gamma^R[\varphi](\vec{\varphi}_1, \vec{\varphi}_2) = \Delta^R_{\mathscr{L}}[\varphi] \circ E''(\mathscr{L})[\varphi](\vec{\varphi}_1, \vec{\varphi}_2) \ .$$

We call $\nabla^R = D + \Gamma^R$ a *retarded Peierls connection.* It is *flat* and *preserves both* $\Delta_{\mathscr{L}|_{\lambda h}}$ *and* $\Delta^H_{\mathscr{L}|_{\lambda h}}$, where the latter is obtained from $\Delta^H_{\mathscr{L}}[\varphi_0]$ as the unique symmetric distributional bisolution of $E'(\mathscr{L}|_{\lambda h})[\varphi]\vec{\varphi} = 0$ which coincides with $\Delta^H_{\mathscr{L}}[\varphi_0]$ on $(\mathscr{M} \smallsetminus J^+(\text{supp } h)) \times (\mathscr{M} \smallsetminus J^+(\text{supp } h))$ [32]. (the above h-cutoffs are always understood around φ_0).

A consequence of the above results is that the following formula holds in $\mathscr{U}$:

$$F \circledast_H G(\varphi) = e^{\hbar \langle \Delta^{+,H}_{\mathscr{L}|_{\lambda h}}[\varphi], \nabla^R_1 \otimes \nabla^R_2 \rangle} F(\varphi_1) G(\varphi_2)|_{\varphi_1 = \varphi_2 = \varphi} \ .$$

The right hand side of the above expression is however, *everywhere* defined on $\mathscr{Q}$. Moreover, since ∇^R is a flat Poisson connection, we conclude that $\circledast_H$ is a *Fedosov-type* *-product on $\mathfrak{A}_{cl}[[\hbar]]$ [21]. This product, however, only contains the "tree level" part of the quantum interaction induced by $\mathscr{L}|_h$. On the other hand, this suggests a path to incorporate the higher order corrections to the interacting *-product. Namely, one should replace $(\mathscr{L}|_h)_{int}$ by a "proper interaction" $\Gamma_R(e^{(\mathscr{L}|_h)_{int}})$ around φ_0 as defined in Appendix A of [3], so that the resulting tree level formulae above yield the full retarded interacting functionals, hence incorporating as well the effects of renormalisation. This represents a rigorous counterpart of the method of *effective action* employed to deal with the loop expansion [16, 40, 49]. The analysis of the convergence of each term in $\hbar$, however, will require careful analysis, possibly requiring methods similar to those employed in [11] to construct classical retarded Møller maps.

b.3. Line of investigation C – applications to condensed matter

We seek now to apply perturbative algebraic quantum field theory to cases which, historically, seemed to be too far both technically and conceptually. However, a recent paper [22], may well result in a strong breakthrough which may lead to substantially enlarge the range of applications of perturbative algebraic quantum field theory. The following description is an elaboration of results found recently in collaboration with Klaus Fredenhagen [8].

b.3.1. Existence of KMS states for BEC

Bose-Einstein condensation has been originally found for noninteracting systems of bosons. A difficult question is whether this effect remains stable under interactions. With the advent of atomic gases of extreme low temperatures states have been experimentally constructed which are interpreted as Bose-Einstein condensates. The theoretical analysis is mainly done within nonrelativistic many body quantum mechanics.

We want to investigate whether the relativistic complex scalar field has states with Bose-Einstein condensates for a local interaction in the sense of formal power series. This may be interpreted as the persistence of condensation for sufficiently small (actually infinitesimal) interaction strength.

KMS states of interacting quantum fields have recently been constructed under rather general conditions [22]. A sufficient condition is the fast decay of truncated expectation values of the corresponding KMS state of the free system.

Let φ denote a free massive complex scalar quantum field. Its equilibrium states with inverse temperature $\beta > 0$ and chemical potential μ, $|\mu| < m$ are quasifree ("Gaussian") states with the 2-point functions

$$\omega_{\beta,\mu}(\varphi^*(x)\varphi(y))) = (2\pi)^{-3} \int d^4p\, \delta(p^2 - m^2)\epsilon(p_0)e^{ip(x-y)} \frac{1}{1 - e^{-\beta(p_0-\mu)}} \tag{10}$$

and $\omega_{\beta,\mu}(\varphi(x)\varphi(y))) = \omega_{\beta,\mu}(\varphi^*(x)\varphi^*(y))) = 0$. They satisfy the KMS condition with respect to the time evolution

$$\alpha_{t,\mu}(\varphi(x)) = \varphi(x + te_0)e^{it\mu} \tag{11}$$

where e_0 denotes the unit vector in time direction.

For chemical potentials $\mu = \pm m$ there exist many equilibrium states at fixed temperature. The pure phases can be obtained from $\omega_{\beta,\pm m}$ by an automorphism which is generated by

$$\gamma_c^{\pm}(\varphi(x)) = \varphi(x) + e^{\pm ix^0 m}c(\mathbf{x}) \tag{12}$$

where c is a harmonic function. The states $\omega^{\pm}_{\beta,c} := \omega_{\beta,\pm m} \circ \gamma_c^{\pm}$ are KMS states w.r.t. $\alpha_{t,\pm m}$ from (11).

An interesting observable for these states is the current density

$$j_0 = -i :\dot{\varphi}^*\varphi - \varphi^*\dot{\varphi}: \tag{13}$$

where the normal ordering is done with respect to the vacuum state $\omega_{\infty,0}$. We find

$$\omega^{\pm}_{\beta,c}(j_0) = \int d^4p\, 2|p_0|\delta(p^2 - m^2) \left(\frac{1}{e^{\beta(|p_0| \mp m)} - 1} - \frac{1}{e^{\beta(|p_0| \pm m)} - 1} \right) \pm 2m|c|^2 \tag{14}$$

Here the first term is the critical charge density at a given temperature, and the second term is the condensate. We observe that at a fixed charge density, there is critical temperature where a condensate is formed.

Let us now introduce an interaction as a gauge invariant polynomial P of φ. According to [22] one can embed the algebra of the interacting theory, restricted to a time slice $-\epsilon < x^0 < \epsilon$ into the algebra of the free theory via a homomorphism γ_χ depending on the choice of a test function χ of time which is equal to 1 on the interval $[-\epsilon, \epsilon]$, and the time evolution of the interacting theory is obtained from that of the free theory by a regularized interaction Hamiltonian density

$$\mathcal{H}_I(\mathbf{x}) = -\int_{-\infty}^{0} dt (P(\varphi))_\chi(t, \mathbf{x})\dot{\chi}(t) \tag{15}$$

where $(P(\varphi))_\chi$ denotes the field $P(\varphi)$ under the interaction which is switched on and off by χ. Due to the smearing in time, $H_I(\mathbf{x})$ is a well defined formal power series of operators within the free theory. For any β-KMS state ω of the free theory and any spatial cutoff by a test function h on $\mathbb{R}^3$ we obtain a KMS state of the theory with interaction $H_I(h) = \int d^3\mathbf{x}\, h(\mathbf{x})\mathcal{H}_I(\mathbf{x})$ by the formula

$$\omega_h(A) = \sum_{n=0}^{\infty} \int_{0 \le \beta_1 \le \ldots \beta_n \le \beta} d\beta_1 \ldots d\beta_n \int_{\mathbb{R}^{3n}} d^3\mathbf{x}_1 \ldots d^3\mathbf{x}_n h(\mathbf{x}_1) \ldots h(\mathbf{x}_n)\, \omega_T(A; \alpha_{i\beta_1}(\mathcal{H}_I(\mathbf{x}_1)); \ldots ; \alpha_{i\beta_n}(\mathcal{H}_I(\mathbf{x}_n)))$$

Here ω_T denotes the truncated functional associated to ω. We observe that for $\beta < \infty$ the limit $h \to 1$ is obtained provided the truncated functions decay sufficiently fast. This is satisfied for $|\mu| < m$.

At zero temperature, all the states with chemical potential $|\mu| < m$ coincide with the vacuum. There the truncated functions decay in $\mathbf{x}_i$ and β_i exponentially fast, so the corresponding ground state exist also for the interacting theory. Now let us look at states with a condensate. Since the vacuum expectation value of the charge density vanishes, any nonvanishing charge density requires a condensate. For definiteness we choose the term $\frac{\lambda}{4}|\varphi|^4$ with $\lambda > 0$ as interaction. One may look at a condensate which is spatially constant and satisfies the field equation with the lowest possible energy density for a given charge density. The field equation then becomes an ordinary differential equation,

$$\ddot{\varphi} + m^2\varphi + \lambda|\varphi|^2\varphi = 0 \ . \tag{16}$$

Using polar coordinates R, θ in the target space we obtain the two conservation laws for this equation in the form

$$j_0 = R^2\dot{\theta} \tag{17}$$

and

$$u = \frac{1}{2}(\dot{R}^2 + R^2\dot{\theta}^2 + m^2R^2) + \frac{1}{4}\lambda R^4 \ . \tag{18}$$

For a given value of $j_0 = \rho_0$, there is unique value $R = R_0 > 0$ for which the effective potential

$$U_{\text{eff}} = \frac{\rho_0^2}{2R^2} + \frac{1}{2}m^2R^2 + \frac{1}{4}\lambda R^4 \tag{19}$$

is minimal. The condensate is then given by

$$\varphi = R_0 e^{i\mu t} \ , \ \mu = \frac{\rho_0}{R_0^2} \ . \tag{20}$$

We now expand the Lagrangian around this configuration. We define a field ψ by

$$\varphi(x) = e^{i\mu x^0}(R_0 + \psi(x)) \tag{21}$$

and obtain, up to total derivatives,

$$\mathcal{L}(\psi, \partial\psi) = \frac{1}{2}\partial\overline{\psi}\partial\psi - \frac{1}{2}(m^2 + 3\lambda R_0^2)\psi_1^2 - \frac{1}{2}(m^2 + \lambda R_0^2)\psi_2^2 - \lambda R_0\psi_1|\psi|^2 - \frac{1}{4}\lambda|\psi|^4 \tag{22}$$

where ψ_1 is the real and ψ_2 the imaginary part of ψ. We split $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ with the free ψ-theory

$$\mathcal{L}_0 = \frac{1}{2}\partial\overline{\psi}\partial\psi - \frac{1}{2}m^2\overline{\psi}\psi \ . \tag{23}$$

In the quantized theory, one has to replace the pointwise products of fields by normal ordered ones, where normal ordering refers to the vacuum of the free φ-theory. For a polynomial in ψ this coincides with the normal ordering with respect to the vacuum of the free ψ theory. We observe that the ψ-theory satisfies the conditions for the existence of ground states and of KMS states, as stated in [22].

It is planned to look at relativistic KMS states, as discussed by Bros and Buchholz [4].

b.3.2. Existence of KMS states for BEC in a trap

Instead of looking at the infinite volume limit one may consider the situation when the condensate is contained in a trap. We simulate the trap by a scalar potential V, such that the Klein-Gordon equation assumes the form

$$(\Box + m^2 + V)\varphi = 0 \ . \tag{24}$$

We choose V such that the operator $K := -\Delta + m^2 + V$ has a single nondegenerate eigenvalue k^2 with $m > k > 0$ and absolutely continuous spectrum $[m^2, \infty)$. Then for chemical potential $|\mu| < k$ the β-KMS state is quasifree with 2-point function

$$\omega_{\beta,\mu}(\varphi(x)^*\varphi(y)) = \left(\frac{1}{2\sqrt{K}}\left(\frac{e^{it\sqrt{K}}}{1 - e^{-\beta(\sqrt{K}-\mu)}} + \frac{e^{-it\sqrt{K}}}{e^{\beta(\sqrt{K}+\mu)} - 1}\right)\right)(\mathbf{x}, \mathbf{y}) \tag{25}$$

where $t = x^0 - y^0$. The expected charge density in this state is

$$\omega_{\beta,\mu}(j_0(\mathbf{x})) = \left(\frac{1}{e^{\beta(\sqrt{K}-\mu)} - 1} - \frac{1}{e^{\beta(\sqrt{K}+\mu)} - 1}\right)(\mathbf{x}, \mathbf{x}) \tag{26}$$

Let c denote the normalized eigenfunction of K with eigenvalue k^2, and let $c_{\mathbf{p}}$ denote the improper eigenfunction with asymptotic behaviour $e^{i\mathbf{px}}$ and the normalization

$$\int d^3\mathbf{x}\overline{c_{\mathbf{p}}(\mathbf{x})}c_{\mathbf{q}}(\mathbf{x}) = \delta(\mathbf{p} - \mathbf{q}) \ . \tag{27}$$

Then

$$\omega_{\beta,\mu}(j_0(\mathbf{x})) = \left(\frac{1}{e^{\beta(k-\mu)}-1} - \frac{1}{e^{\beta(k+\mu)}-1}\right)|c(\mathbf{x})|^2 \tag{28}$$

$$+\int d^3\mathbf{p}\left(\frac{1}{e^{\beta(\sqrt{\mathbf{p}^2+m^2}-\mu)}-1} - \frac{1}{e^{\beta(\sqrt{\mathbf{p}^2+m^2}+\mu)}-1}\right)|c_p(\mathbf{x})|^2\ . \tag{29}$$

We observe that the charge density diverges as $\mu \uparrow k$ and that it is dominated by the contribution of the eigenfunction c,

$$(k-\mu)\omega_{\beta,\mu}(j_0(\mathbf{x})) \to \beta^{-1}|c(\mathbf{x})|^2\ . \tag{30}$$

Since the contribution of the continuous spectrum decays as $\beta^{-\frac{3}{2}}e^{-\beta(m-k)}$, one observes a rather sharp transition at a critical temperature, but not a phase transition. The KMS states is expected to satisfy exponential clustering, hence also the KMS states for interacting theory will exist. Details will be soon available.

b.4. Line of investigation D – Quantum Gravity and Cosmology

In the last stage we would like to put together a good deal of the above results in order to address the most outstanding open problem in quantum field theory, namely the quantisation of gravity.

b.2.2 Perturbative Quantum Gravity

The last decades were dominated by attempts to reach this goal by rather radical new concepts, the best known being string theory and loop quantum gravity. A more conservative approach via quantum field theory was originally considered to be hopeless because of severe conceptual and technical problems. In the meantime it became clear that the first two attempts also meet enormous problems, and it might be worthwhile to make a reappraisal of the quantum field theoretical approach. In fact, there are indications that the obstacles in this approach are not as insurmountable as originally expected.

One of these obstacles is *perturbative non-renormalizability*, that is, the counterterms arising in higher orders of perturbation theory cannot be taken into account by redefining the parameters of the gravitational Lagrangian. Nevertheless, even theories with this property may be considered as *effective theories*, with the property that only finitely many renormalization parameters need to be considered below a fixed energy scale [49]. Moreover, it may be that the theory is actually *asymptotically safe*, in the sense that there is an ultraviolet fixed point of the renormalisation group flow with only finitely many relevant directions [35].

Another obstacle is the incorporation of the principle of general covariance. Quantum field theory is traditionally based on the isometry group of Minkowski space-time, the Poincar group. In particular, concepts such as vacuum states, asymptotic particle states and the scattering matrix heavily rely on Poincar symmetry. Quantum field theory on curved space-times, which might be considered as an intermediate step towards quantum gravity, already lacks a particle interpretation due to the generic absence of nontrivial space-time isometries. In fact, one of the most spectacular results of quantum field theory on curved spacetimes is Hawking's prediction of black hole evaporation [47], a result which may be understood as a consequence of different particle interpretations in different space-time regions.

Our goal was to to extend the approach outlined in the previous sections also to gravity. Unfortunately, here we meet, besides the aforementioned difficulties, a major conceptual stumbling block. To wit, a successful treatment of quantum field theory on generic space-times requires the use of local observables, but unfortunately there are no *diffeomorphism invariant* localized functionals of the dynamical degrees of freedom – in the case of (pure) gravity, the metric g itself. The way out is to replace the requirement of invariance by *covariance*, which amounts to consider *relative* or *partial observables* in the sense of [41]. Whereas the classical meaning of these concepts is essentially clear, the corresponding concepts in quantum theory pose conceptual and technical problems. Our recent paper [9] contains results that give a positive answer to those questions. Moreover, we have been able to address positively the most complicated conceptual issue of *background independence*, which constitute a strong *desideratum* for perturbative quantum gravity. Eventually, the fulfilment of the strategy outlined in the previous sections should lead naturally to the possibility of doing perturbation theory also around *any* classically interacting degrees of freedom, which is one of our main future tasks. . We expect that the deformation quantisation strategy outlined in Subsection b.2.1 should prove effective in addressing this issue in the present context as well.

Clearly, since a perturbative construction of *pure* quantum gravity has been achieved, one may aim far higher and try to pursue other directions of investigation.

b.2.3 Perturbative Quantum Gravity and matter

One such direction concerns the effects of quantum gravity on matter. There has been recent theoretical work [45] indicating that models lacking asymptotic freedom in the ultraviolet regime, such as quantum electrodynamics (QED), could actually acquire this property after the incorporation of quantum gravitational effects. This qualitative change in the high energy behaviour of QED seems to come from the incorporation of screening effects coming from gravity at very small scales. The validity of this claim, however, is still a matter of debate in the literature. Due to the importance of this phenomenon, we would like to see if within our conceptually clear and technically precise framework we could verify its actual occurrence.

b.2.2 Perturbative Quantum Gravity and Cosmology

The fluctuations of the cosmological microwave background provide a deep insight into the early history of the universe. The most successful theoretical explanation is inflationary cosmology where a scalar field (the inflaton) is coupled to the gravitational field. Usually, the theory is considered in linear order around a highly symmetric background, typically the spatially flat Friedmann-Lemaitre-Robertson-Walker spacetime. Extending the theory to higher orders is accompanied by severe obstacles. Already in a classical analysis the definition of gauge invariant observables turns out to be rather complicated; moreover, one is immediately confronted with the problem of constructing a theory of quantum gravity.

A possible solution relies on the previously recalled construction of relative observables and works as follows. One selects 4 scalar fields $X^a_\Gamma, a = 1, \dots 4$, which are local functionals of the field configuration Γ which includes the spacetime metric g, the inflaton field ϕ and possibly other fields. The fields X^a_Γ are supposed to transform under diffeomorphisms χ as

$$X^a_{\chi^*\Gamma} = X^a_\Gamma \circ \chi \ . \tag{31}$$

We choose a background Γ_0 such that the map

$$X_{\Gamma_0} : M \ni x \mapsto (X^1_{\Gamma_0}, \dots, X^4_{\Gamma_0}) \in \mathbb{R}^4 \tag{32}$$

is injective. We then set for Γ sufficiently near to Γ_0

$$\alpha_\Gamma = X^{-1}_\Gamma \circ X_{\Gamma_0} \tag{33}$$

We observe that α_Γ transforms under diffeomorphisms as

$$\alpha_{\chi^*\Gamma} = \chi^{-1} \circ \alpha_\Gamma \tag{34}$$

Let now A_Γ be any other scalar field which is a local functional of Γ and transforms under diffeomorphisms as in (31). Then the field

$$\mathcal{A}_\Gamma \doteq A_\Gamma \circ \alpha_\Gamma \tag{35}$$

is invariant under diffeomorphisms and may be considered as a local observable. Note that invariance is obtained by shifting the argument of the field in a way which depends on the configuration.

One has, of course, to be careful in the characterization of the region where all the maps are well defined and one has to restrict oneself to configurations Γ in some neighborhood of the background Γ_0. Hence the verdict that local observables do not exist in a diffeomorphism invariant theory remains true if one wants to consider them as functionals on the full configuration space. But locally in configuration space, local observables exist as shown above.

Fortunately, in formal deformation quantization as well as in perturbation theory, only the Taylor expansion of observables at the background configuration enters, hence it suffices to establish the injectivity of X_{Γ_0}. As an example we compute the expansion up to first order. We obtain

$$\mathcal{A}_{\Gamma_0+\delta\Gamma} = A_{\Gamma_0} + \left\langle \frac{\delta A_\Gamma}{\delta\Gamma}(\Gamma_0), \delta\Gamma \right\rangle + \frac{\partial A_{\Gamma_0}}{\partial x^\mu} \left\langle \frac{\delta \alpha^\mu_\Gamma}{\delta\Gamma}(\Gamma_0), \delta\Gamma \right\rangle \ . \tag{36}$$

The last term on the right hand side is necessary in order to get gauge invariant fields (up to 1st order). We calculate

$$\frac{\delta \alpha^\mu_\Gamma}{\delta\Gamma}(\Gamma_0) = -\left(\left(\frac{\partial X_{\Gamma_0}}{\partial x} \right)^{-1} \right)^\mu_a \frac{\delta X^a_\Gamma}{\delta\Gamma}(\Gamma_0) \ . \tag{37}$$

We can now apply the general theory to inflationary cosmology. Progress in this direction is quick. We expect to put forward our results by the end of March 2016.

Once this is done, clearly perturbative quantum gravity in the locally covariant approach has standard inflationary cosmology as a first order approximation. All that by crystal clear physical principles and rigorous technical arguments. What can be done next? A first important direction is certainly the computation of the spectrum and bispectrum of the background radiation. We expect no conceptual difficulties in the practical computations. A much more ambitious task would be the study of the problem of black-holes evaporation.

References

[1] D. Bahns, K. Rejzner, J. Zahn, *The Effective Theory of Strings.* `arXiv:1204.6263 [math-ph]`.

[2] F. Bayen, M. Flato, C. Frønsdal, A. Lichnerowicz, D. Sternheimer, *Deformation Theory and Quantization. I. Deformations of Symplectic Structures; II. Physical Applications.* Ann. Phys. **111** (1978) 61–110 (I); 111–151 (II).

[3] F. Brennecke, M. Dütsch, *Removal of Violations of the Master Ward Identity in Perturbative QFT.* Rev. Math. Phys. **20** (2008) 119–172. `arXiv:0705.3160 [hep-th]`.

[4] J. Bros, D. Buchholz, *Towards a relativistic KMS condition.* Nucl. Physics B. **429** (1994) 91.

[5] R. Brunetti, M. Dütsch, K. Fredenhagen, *Perturbative Algebraic Quantum Field Theory and the Renormalization Groups.* Adv. Theor. Math. Phys, to appear (2009). `arXiv:0901.2038 [math-ph]`.

[6] R. Brunetti, K. Fredenhagen, *Microlocal Analysis and Interacting Quantum Field Theories: Renormalization on Physical Backgrounds.* Commun. Math. Phys. **208** (2000) 623–661. `arXiv:math-ph/9903028`.

[7] R. Brunetti, K. Fredenhagen, *Quantum Field Theory on Curved Backgrounds.* Lecture notes, Kompaktkurs "Quantenfeldtheorie auf gekrümmten Räumzeiten"(C. Bär, K. Fredenhagen, orgs.), Universität Potsdam, October 8th–12th 2007, `arXiv:0901.2063 [math-ph]`.

[8] R. Brunetti, K. Fredenhagen, *Bose-Einstein condensation for relativistic Bose gas*, (2014) manuscript.

[9] R. Brunetti, K. Fredenhagen, K. Rejzner, *Quantum Gravity as a Locally Covariant Field Theory.* Commun. Math. Physics, (2016) in course of publication.`arXiv:1306.1058 [math-ph]`.

[10] R. Brunetti, K. Fredenhagen, P. L. Ribeiro, *Algebraic Structure of Classical Field Theory I: Kinematics and Linearized Dynamics for Real Scalar Fields.* `arXiv:1209.2148 [math-ph]`.

[11] R. Brunetti, K. Fredenhagen, P. L. Ribeiro, *Algebraic Structure of Classical Field Theory II: The Main Dynamical Theorem and its Applications.* To appear.

[12] R. Brunetti, K. Fredenhagen, P. L. Ribeiro, *The Principle of Space-Time Descent.* Poster presentation at the XVII International Congress on Mathematical Physics (Aalborg, August 6–11, 2012). Available at `http://docs.math.aau.dk/timeline/e-posters/S09/Ribeiro.pdf`.

[13] R. Brunetti, K. Fredenhagen, R. Verch, *The Generally Covariant Locality Principle: a New Paradigm for Local Quantum Physics.* Commun. Math. Phys. **237** (2003) 31–68. `arXiv:math-ph/0112041`.

[14] R. Brunetti, P. L. Ribeiro, *A Semiclassical Interacting *-Product in Perturbative Algebraic Quantum Field Theory.* Oral presentation at the Conference "Mathematical Aspects of Quantum Field Theory and Quantum Statistical Mechanics" (Hamburg, July 30th – August 1st, 2012). Available at `http://www.qq12.org/talks/Monday/8.Ribeiro.pdf`.

[15] A. Cannas da Silva, A. Weinstein, *Geometric Models for Noncommutative Algebras* (American Mathematical Society, 1999).

[16] B. S. DeWitt, *The Global Approach to Quantum Field Theory – Volume 1* (Oxford University Press, 2003).

[17] G. Dito. *Deformation Quantization of Covariant Fields.* In: G. Halbout (ed.), *Deformation Quantization: Proceedings of the meeting between mathematicians and theoretical physicists (Strasbourg, 2001).* IRMA Lectures in Math. Theoret. Phys. **1** (Walter De Gruyter, 2002), pp. 55–66. `arXiv:math/0202271`.

[18] M. Dütsch, K. Fredenhagen, *Algebraic Quantum Field theory, Perturbation Theory, and the Loop Expansion.* Commun. Math. Phys **219** (2001) 5–30. `arXiv:hep-th/0001129`.

[19] M. Dütsch, K. Fredenhagen, *The Master Ward Identity and Generalized Schwinger-Dyson Equation in Classical Field Theory.* Commun. Math. Phys **243** (2003) 275–314. `arXiv:hep-th/0211242`.

[20] H. Epstein, V. Glaser, *The Role of Locality in Perturbation Theory.* Ann. Inst. H. Poincar Sect. A Phys. Thor. **19** (1973) 211–295.

[21] B. V. Fedosov, *A Simple Geometrical Construction of Deformation Quantization.* J. Diff. Geom. **40** (1994) 213–238.

[22] K. Fredenhagen, F. Lindner, . Commun. Math. Phys. **332** (2014) 895-932. `arXiv:1306.6519 [math-ph]`.

[23] K. Fredenhagen, K. Rejzner, *Batalin-Vilkovisky Formalism in the Functional Approach to Classical Field Theory.* Commun. Math. Phys. **314** (2012) 93–127. `arXiv:1101.5112 [math-ph]`.

[24] S. A. Fulling. F. J. Narcowich, R. M. Wald, *Singularity Structure of the Two-Point Function in Quantum Field Theory in Curved Spacetime. II.* Ann. Phys. **136** (1981) 243–272.

[25] S. A. Fulling. M. Sweeny, R. M. Wald, *Singularity Structure of the Two-Point Function in Quantum Field Theory in Curved Spacetime.* Commun. Math. Phys. **63** (1978) 257–264.

[26] D. Giulini, E. Joos, C. Kiefer, J. Kupsch, I.-O. Stamatescu, H. D. Zeh, *Decoherence and the Appearance of a Classical World in Quantum Theory.* Springer-Verlag, 1996.

[27] J. Glimm, A. Jaffe, *Quantum Physics – a Functional Integral Approach (Second Edition)* (Springer-Verlag, 1987).

[28] R. Haag, *Local Quantum Physics: Fields, Particles, Algebras.* 2nd. Ed., Springer-Verlag, 1996.

[29] M. Henneaux, C. Teitelboim, *Quantization of Gauge Systems.* Princeton University Press, 1994.

[30] K. Hepp, *The Classical Limit for Quantum Correlation Functions.* Commun. Math. Phys. **35** (1974) 265–277.

[31] S. Hollands, *Renormalized Quantum Yang-Mills Fields in Curved Spacetime.* Rev. Math. Phys. **20** (2008) 1033–1172. arXiv:0705.3340 [gr-qc].

[32] S. Hollands, R. M. Wald, *Conservation of the Stress Tensor in Perturbative Interacting Quantum Field Theory in Curved Spacetimes* . Rev. Math. Phys **17** (2005) 227–312. arXiv:gr-qc/0404074.

[33] D. Joyce, *An Introduction to $\mathscr{C}^\infty$ Schemes and $\mathscr{C}^\infty$ Algebraic Geometry.* Surv. Diff. Geom. **17** (2012) 299–325. arXiv:1001.0023 [math].

[34] M. Kontsevich, *Deformation Quantization of Poisson Manifolds.* Lett. Math. Phys. **66** (2003) 157–216. arXiv:q-alg/9709040.

[35] R. Percacci, *Asymptotic Safety.* In: "Approaches to Quantum Gravity: Towards a New Understanding of Space, Time and Matter" (D. Oriti, Ed.), Cambridge University Press, 2009. arXiv:0709.3851 [hep-th].

[36] M. J. Radzikowski, *The Hadamard Condition and Kay's Conjecture in (Axiomatic) Quantum Field Theory on Curved Space-Time.* Ph.D. Thesis, Princeton University, 1992.

[37] K. Rejzner, *Fermionic Fields in the Functional Approach to Classical Field Theory.* Rev. Math. Phys. **23** (2011) 1009–1033. arXiv:1101.5126 [math-ph].

[38] K. Rejzner, *Batalin-Vilkovisky formalism in Locally Covariant Field Theory.* PhD Thesis, Hamburg University, 2011.

[39] M. A. Rieffel, *Deformation Quantization for Actions of* $\mathbb{R}^d$. Mem. Amer. Math. Soc. **106** (1993), 93 pp.

[40] R. J. Rivers, *Path Integral Methods in Quantum Field Theory.* Cambridge University Press, 1988.

[41] Rovelli, *Partial Observables.* Phys.Rev.D65:124013, 2002.

[42] G. Scharf, *Finite Quantum Electrodynamics: The Causal Approach.* 2nd. Ed., Springer-Verlag, 1995.

[43] I. M. Singer, *Some Remarks on the Gribov Ambiguity.* Commun. Math. Phys. **60** (1978) 7–12.

[44] S. J. Summers, *A Perspective on Constructive Quantum Field Theory.* Article commissioned for UNESCO's Encyclopedia of Life Support Systems (EOLSS – http://www.eolss.net). arXiv:1203.3991 [math-ph].

[45] D.J. Toms, *Quantum Gravitational Contributions to Qauntum Electrodynamics.* Nature 468:56-59, 2010.

[46] A. Vistoli, *Notes on Grothendieck Topologies, Fibered Categories and Descent Theory.* In: B. Fantechi *et al.*, *Fundamental Algebraic Geometry: Grothendieck's FGA Explained* (American Mathematical Society, 2005), pp. 1–104. Updated version (2008): http://homepage.sns.it/vistoli/descent.pdf

[47] R. M. Wald, *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics.* Chicago University Press, 1994.

[48] S. Waldmann, *Poisson-Geometrie und Deformationsquantisierung.* Springer-Verlag, 2007.

[49] S. Weinberg, *The Quantum Theory of Fields, Volume II: Modern Applications.* Cambridge University Press, 1996.